Model Categories by Example

Lecture 5

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Recap

* $X, Y \in \mathcal{C} \rightsquigarrow \mathrm{map}(X, Y) \in sSet$ Kan complex. "Reedy resolutions"
* Left Bousfield localization. $\mathcal{C}$ a model cat. S a <u>set</u> of maps

$\rightsquigarrow$ S-local objects + S-local equivalences

$L_S \mathcal{C}$

W = S-local equivs $\supseteq$ weak equivalences of $\mathcal{C}$

F = RLP (acyclic cof)

C = Cofib. of $\mathcal{C}$.

Recap

Showed Left Bousfield loc^n need not exist in general.

Motivated via example of Barton
$$\begin{tikzcd} \bullet \arrow[r, two heads, "\sim"] \arrow[d, hook, "\sim"'] & \bullet \arrow[d, two heads] \\ \bullet \arrow[r, hook, two heads] & \bullet \end{tikzcd} \Rightarrow \; !$$

$\Rightarrow$ Required "left properness"

LBL exists under the assumption of left prop + combinatorial (cellular)

<u>Ex</u> $Ch(\mathbb{Z})_{proj}$

Form a LBL at those maps f such that $f \otimes \mathbb{Z}/p$ is a w.e.

$\Rightarrow$ derived p-complete ab. grps.

Right Bousfield localization

Let $\mathcal{C}$ be a model category and K a set of objects of $\mathcal{C}$.

- A morphism $f\colon A \to B$ in $\mathcal{C}$ is a *K-coequivalence* if

$$\mathrm{map}(X, f)\colon \mathrm{map}(X, A) \to \mathrm{map}(X, B)$$

  is a weak equivalence in $\mathbf{sSet}_{\mathrm{Kan}}$ for each $X \in K$.

- An object $Z \in \mathcal{C}$ is *K-colocal* if

$$\mathrm{map}(Z, f)\colon \mathrm{map}(Z, A) \to \mathrm{map}(Z, B)$$

  is a weak equivalence in $\mathbf{sSet}_{\mathrm{Kan}}$ for any K-coequivalence.

Right Bousfield localization

The *right Bousfield localization* at K of $\mathcal{C}$ is the model category $R_K\mathcal{C}$ with underlying category of $\mathcal{C}$ such that the:

- weak equivalences are K-colocal equiv $\supseteq$ weak equiv of $\mathcal{C}$.
- fibrations are the fib. of $\mathcal{C}$

$id: \mathcal{C} \to R_K\mathcal{C}$ right Quillen.

- cofibrations LLP(acyclic fib)

$Cell_K\mathcal{C}$ K-cell-$\mathcal{C}$

Exists for $\mathcal{C}$ comb/cellular + right properness.

Torsion objects

Return to $Ch(\mathbb{Z})_{proj}$. (is left + right proper)

Perform RBL at $\mathbb{Z}/p$.

$R_{\mathbb{Z}/p} Ch(\mathbb{Z}) \rightsquigarrow$ derived p-torsion abelian groups

$$\mathrm{Id} : R_{\mathbb{Z}/p} Ch(\mathbb{Z})_{proj} \underset{\simeq_Q}{\rightleftarrows} L_{\mathbb{Z}/p} Ch(\mathbb{Z})_{proj} : \mathrm{Id}$$

Preservation of properties

<u>Fact</u> Localization is agressive.

* Take $\mathcal{C}$, and consider $L_S \mathcal{C}$.

  I, J $\qquad$ I but finding a J here is $\approx$ impossible.

* If $\mathcal{C}$ was right proper, $L_S \mathcal{C}$ has no reason to be right proper.
* would like conditions on $\mathcal{C}, S$ to preserve some of these things.

  * Investigate Monoidal Model categories.

Monoidal model categories

Idea: If $\mathcal{C}$ is a (symmetric) monoidal category $(\otimes, \mathbb{1})$, would like it to interact with $\mathrm{Ho}(\mathcal{C})$

In general if

$$\mathcal{C} \text{ monoidal} + \text{a model cat} \not\Rightarrow \mathrm{Ho}(\mathcal{C}) \text{ is monoidal}$$

A sufficient condition for this $\rightsquigarrow$ monoidal model cat.

Monoidal model categories

A *symmetric monoidal model category* is a model category $\mathcal{C}$ equipped with a closed symmetric monoidal structure $(\mathcal{C}, \otimes, \mathbb{1})$ such that the two following compatibility conditions are satisfied:

(1) (Pushout-product axiom) For every pair of cofibrations $f\colon X \to Y$ and $f'\colon X' \to Y'$, their *pushout-product*

$$f \square f'\colon (X \otimes Y') \coprod_{X \otimes X'} (Y \otimes X') \to Y \otimes Y'$$

if $X \simeq 0$

$\Rightarrow X \otimes X \simeq 0$

is also a cofibration. Moreover, it is a acyclic cofibration if either f or f' is.

(2) (Unit axiom) For every cofibrant object X and every cofibrant resolution $\varnothing \hookrightarrow Q\mathbb{1} \overset{\sim}{\twoheadrightarrow} \mathbb{1}$ of the tensor unit, the induced morphism $Q\mathbb{1} \otimes X \to \mathbb{1} \otimes X$ is a weak equivalence.

Monoidal model categories

Proposition: Let $\mathcal{C}$ be a closed monoidal model category. Then $Ho(\mathcal{C})$ is also closed monoidal with tensor structure

$$(\otimes^{L}, R\mathbb{1})$$

P.P $\ni$ $\otimes : \mathcal{C} \times \mathcal{C} \to \mathcal{C}$ left Quillen.

Examples

* $sSet_{Kan}$ + $sSet_{Joyal}$ both MMC w.r.t $\times$

* Cat_{Nat} w.r.t $\times$ is MMC.

* Cat_{Thom} is <u>not</u> MMC (Raptis "Homotopy theory of Posets" §3)

* $Top_{Quillen}$ is <u>not</u> MMC

  Fix: Restrict to a "convenient subcat of Top"

Examples

* If R is a comm. ring then $Ch(R)_{proj}$ is monoidal model cat w.r.t $\otimes$-product of chain complexes

! $Ch(R)_{inj}$ is rarely monoidal!

$\Rightarrow$ being a monoidal model cat is not preserved under Quillen equivalence.

Ex $R = \mathbb{Z}$ $\quad \mathbb{Z} \hookrightarrow \mathbb{Q} \quad 0 \hookrightarrow \mathbb{Z}/2\mathbb{Z}$

$$(\mathbb{Z} \hookrightarrow \mathbb{Q}) \square (0 \hookrightarrow \mathbb{Z}/2\mathbb{Z}) = \mathbb{Z}/2\mathbb{Z} \to 0$$

not an injective cofibration.

Arrow categories

Let $\mathcal{C}$ be combinatorial model

$I = (\bullet \to \bullet) \qquad \mathrm{Arr}(\mathcal{C}) = \mathcal{C}^I.$

Assume moreover that $\mathcal{C}$ is symmetric monoidal model cat $(\otimes, \mathbb{1})$.

Idea: $\mathrm{Arr}(\mathcal{C})$ has 2 different monoidal structures playing nicely with proj/inj model.

Arrow categories

The *tensor product monoidal structure* on **Arr**$(\mathcal{C})$ is defined as

$$X_0 \otimes Y_0 \xrightarrow{f \otimes g} X_1 \otimes Y_1$$

for morphisms $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$. The monoidal unit in this structure is $\mathbb{1} \to \mathbb{1}$.

Prop: $\mathrm{Arr}(\mathcal{E})_{inj}$ is symm. monoidal model cat w.r.t the tensor product monoidal structure.

But $\mathrm{Arr}(\mathcal{E})_{proj}$ NOT.

Arrow categories

The *pushout-product monoidal structure* on **Arr**$(\mathcal{C})$ is defined by the pushout-product

$$(X_0 \otimes Y_1) \coprod_{X_0 \otimes Y_0} (X_1 \otimes Y_0) \xrightarrow{f \square g} X_1 \otimes Y_1$$

White - Yau

for morphisms $f\colon X_0 \to X_1$ and $g\colon Y_0 \to Y_1$. The monoidal unit in this structure is $\varnothing \to \mathbb{1}$.

Prop: $\mathrm{Arr}(\mathcal{E})_{proj}$ is a SMMC w.r.t this pushout-product monoidal structure

$\mathcal{E} = \mathrm{sSet}$ $\quad \mathrm{sSet}^{I}$

! Not w.r.t $\mathrm{Arr}(\mathcal{E})_{inj}$

Monoidality from homotopy

<u>Recall</u> Can have $Ho(\mathcal{E}) \simeq Ho(D)$ without $\mathcal{E} \simeq_Q D$

There exists a model cat odSet (open dendroidal sets)

which has a $\otimes$, but is <u>not</u> part of a closed monoidal structure.

$\Rightarrow$ odSet not a monoidal Model cat ($\otimes$ is associative upto weak Equiv)

Cisinski-Moerdijk $\rightsquigarrow$ $Ho(\text{odSet})$ is symmetric monoidal Cat.

Left Bousfield localization .vs. monoidal structures

R-mod$_{st}$ R a Frobenius ring. If R is a f.d Hopf algebra / $\mathbb{K}$ then R-mod$_{st}$ is a monoidal model cat w.r.t $- \otimes_{\mathbb{K}} -$

White

Let $R = \mathbb{F}_2[\Sigma_3]$. An R-module is an $\mathbb{F}_2$-vector space with an action of Σ_3.

R is f.d Hopf algebra / $\mathbb{F}_2$.

$\Rightarrow$ R-mod$_{st}$ is a monoidal model cat.

Left Bousfield localization .vs. monoidal structures

Localise at $f: \mathbb{O} \to \mathbb{F}_2$. (codomain has trivial $\mathcal{E}_3$ action)

<u>Claim</u> $L_f R\text{-mod}_{st}$ is <u>not</u> a MMC (P.P axiom fails)

* Enough to find an f-trivial object N such that $N \otimes_{\mathbb{F}_2} N$ is <u>not</u> f-trivial.

* N f-trivial iff it has no $\mathcal{E}_3$-fixed points

<u>Def^n</u> N is f-trivial if $N \simeq \mathbb{O}$ in $L_f R\text{-mod}$.

Left Bousfield localization .vs. monoidal structures

Let $N = \mathbb{F}_2 \oplus \mathbb{F}_2$. $\quad a = (1,0) \quad b = (0,1)$

$(12)a = b \quad (123)a = b$

N has no Σ_3-fixed points $\quad (123)b = a+b =: c$

$\Rightarrow$ f-trivial

$N \otimes_{\mathbb{F}_2} N$ has Σ_3-invariant element $a \otimes a + b \otimes b + c \otimes c$.

$\Rightarrow$ $N \otimes_{\mathbb{F}_2} N$ is <u>not</u> f-trivial!

$\Longrightarrow$ $L_f R\text{-mod}_{S^1}$ is <u>not</u> a MMC

A sufficient condition

Let $\mathcal{C}$ be a monoidal model category. A left Bousfield localization of $\mathcal{C}$ is said to be *monoidal* if $L_S\mathcal{C}$ is a monoidal model category and the Quillen functor $\mathcal{C} \to L_S\mathcal{C}$ is monoidal.

Let $\mathcal{C}$ be a monoidal model category, then we say that a cofibrant object $X \in \mathcal{C}$ is *flat* if $f \otimes X$ is a weak equivalence whenever f is a weak equivalence.

In above example all X are cofibrant

all X are flat.

A sufficient condition

Proposition: Let $\mathcal{C}$ be a cofibrantly generated monoidal model category in which all cofibrant objects are flat, then a left Bousfield localization $L_S\mathcal{C}$ is a monoidal Bousfield localization if and only if every morphism of the form $f \otimes \mathrm{id}_K$ is an S-local equivalence where $f \in S$ and K cofibrant.

In the previous example $f : 0 \to \mathbb{F}_2$

...and the rest

* Stability

⋮

Lots more to discover!

...and the rest